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Cycle-balance conditions for distance-regular graphs

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Abstract

In a distance-regular graph, the partition with respect to distance from a vertex supports a unique eigenvector for each eigenvalue. There may be non-singleton vertex sets whose corresponding distance partition also supports eigenvectors. We consider the members of three families of distance regular graphs, the Johnson Graphs, Hamming Graphs and Complete Multipartite graphs. For each we determine all such sets which support an eigenvector for the next to largest eigenvalue. These sets exhibit the underlying geometric structure of the graph.

Keywords: Distance-regular graph; Vertex partition; Geometry

1. Introduction and definitions

Let $\mathscr{V}$ be the vertex set of a connected graph Γ, $\mathbf{u}\in\mathbb{R}^{\mathscr{V}}$ be a nonzero vector with entries indexed by $\mathscr{V}, D$ be a nonempty subset of $\mathscr{V}$, and $\pi=[C_0,C_1,\ldots,C_\rho]$ be an ordered partition of $\mathscr{V}$. We say that π *supports* $\mathbf{u}$ if vertices in the same cell always index entries of equal value. We say that π *strongly supports* $\mathbf{u}$ if the entries of $\mathbf{u}$ take $\rho+1$ distinct values and C_i is the set on which the ith from largest value is taken. We say that π is a *path partition* if there are no edges from C_i to C_j when $|i-j|>1$. The *distance partition of* $\mathscr{V}$ *with respect to* D is the path partition $\pi_D=[C_0,C_1,\ldots,C_\rho]$ whose cells are

$$C_i=\{\gamma\in\mathscr{V}\mid \operatorname{dist}(\gamma,D)=i\}.$$

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We say that D *supports* (strongly supports resp.) $\mathbf{u}$ if π_D supports (strongly supports resp.) $\mathbf{u}$. We call π *equitable* if there are $(\rho+1)^2$ constants s_{ij} such that each vertex in C_i has exactly s_{ij} neighbors in C_j. The subset D is said to be *completely regular* if π_D is equitable.

All named vectors are assumed to be *nonzero* although we do not always repeat this assumption. Similarly, any named subset is assumed to be *nonempty*. For $\mathbf{u}\in\mathbb{R}^{\mathscr{V}}$ we write $\mathbf{u}(\gamma)$ for the entry indexed by $\gamma\in\mathscr{V}$, $\langle\mathbf{u}\rangle$ for the one-dimensional subspace spanned by $\mathbf{u}$ and $\bar{\mathbf{u}}=\mathbf{u}/|\mathbf{u}|$ for the unit vector in the direction of $\mathbf{u}$. The reverse of $\pi=[C_0,C_1,\ldots,C_\rho]$ is $\pi^*=[C_\rho,\ldots,C_1,C_0]$.

Every graph Γ in this paper will be a simple regular connected graph. In addition, the graphs treated in the main theorems are *distance-regular*; each singleton vertex set is completely regular and the numbers s_{ij} arising from the corresponding equitable partition do not depend on the particular vertex. Completely regular sets are useful in the study of distance-regular graphs because an equitable distance partition of $\mathscr{V}$ supports eigenvectors for exactly $\rho+1$ eigenvalues of the adjacency matrix of Γ.

Three families of distance-regular graphs are the Johnson Graphs $J(n,d)$, the Hamming Graphs $H(d,n)$, and the Complete Multipartite Graphs $K_{d\times n}$. The main result of this paper is to determine, for each member Γ of these three families, all nonempty vertex sets, all path partitions and all nonzero eigenvectors of certain types as well as some strong relations between them. Let V_1 be the eigenspace for θ_1, the next to the largest eigenvalue of Γ. We will determine

(a) All $\mathbf{u}\in V_1$ which satisfy the *cycle-balance conditions* (B3) and (B4) given below.
(b) All pairs $(\pi,\mathbf{u})$ such that π is a path partition, $\mathbf{u}\in V_1$, and π supports $\mathbf{u}$.

Note that (b) gives all path partitions, distance partitions and equitable distance partitions which support some $\mathbf{u}\in V_1$ and thus all vertex sets and all completely regular vertex sets which support some $\mathbf{u}\in V_1$. It will turn out that everything follows easily from (a). The statement and proofs of the main theorems are similar, although not enough so to be combined into a single theorem or proof. The theorems are stated in the next section. The common material has been gathered as a series of lemmas in Section 3. Then each of the three families is treated in a shorter section. These sections have been written to emphasize the similarities. The final section compares the three cases and gives some general comments.

Given any $\mathbf{u}\in\mathbb{R}^{\mathscr{V}}$, let the *type* of an oriented edge (γ,δ) be the ordered pair $(\mathbf{u}(\gamma),\mathbf{u}(\delta))$. Say that a cycle in Γ is *balanced with respect to* $\mathbf{u}$ if, given a cyclic orientation, it has as many edges of type (a,b) as of type (b,a) where (a,b) ranges over all ordered pairs of reals. We will only consider these two *cycle-balance conditions* on $\mathbf{u}$:

$$\text{Every 3-cycle of } \Gamma \text{ is balanced with respect to } \mathbf{u}. \tag{B3}$$

$$\text{Every 4-cycle of } \Gamma \text{ is balanced with respect to } \mathbf{u}. \tag{B4}$$

If $\mathbf{u}$ is supported by a path partition π, then every cycle in Γ will be balanced with respect to $\mathbf{u}$. This is because an oriented cycle must have as many edges going from C_i to C_{i+1} as from C_{i+1} to C_i.

2. Main theorems

Let Γ be a regular simple connected graph with vertex set $\mathscr{V}$ and adjacency matrix A. The largest eigenvalue of A is the common vertex degree. Let $V_1 \subset \mathbb{R}^{\mathscr{V}}$ denote the eigenspace for the next to largest eigenvalue of A.

To keep the statements of the theorems short, it is convenient to make another definition. An *exclusive pair* $(\pi, \mathbf{u})$ consists of a path partition $\pi = (C_0, \dots, C_\rho)$ and a vector $\mathbf{u} \in V_1$ such that

1. π strongly supports $\mathbf{u}$.
2. $\{\mathbf{w} \in V_1 \mid \pi \text{ supports } \mathbf{w}\} = \langle \mathbf{u} \rangle$.
3. The only other path partition which supports $\mathbf{u}$ is π^*.

When $(\pi_D, \mathbf{u})$ is an exclusive pair we will also say that $(D, \mathbf{u})$ is an exclusive pair.

Remark. $(\pi, \mathbf{u})$ is an exclusive pair if and only if $(\pi^*, -\mathbf{u})$ is. Then the only sets which might support $\mathbf{u}$ are C_0 and/or C_ρ (depending on π and/or π^* being distance partitions.)

The Johnson Graph $J(n,d)$ has as vertices the d-sets of an n-set X and as edges all vertex pairs (γ, δ) such that $\gamma \cap \delta$ has cardinality $d-1$. For $\emptyset \subset Y \subset X$, let the *shadow* of Y be $D_Y = \{\gamma \in \mathscr{V} \mid \gamma \subseteq Y \text{ or } Y \subseteq \gamma\}$ and write π_Y for the distance partition π_{D_Y} with respect to D_Y.

Theorem J. *Let $\mathscr{V} = \{\gamma \subset X \mid |\gamma| = d\}$ be the vertex set of $J(n,d)$.*

(1) *For each $\emptyset \subset Y \subset X$, the shadow D_Y strongly supports a vector $\mathbf{u}_Y \in V_1$.*
(2) *Each D_Y is completely regular.*
(3) *Suppose $\mathbf{u} \in V_1$ satisfies* (B3). *Then the set on which it takes its maximum is a shadow D_Y and $\bar{\mathbf{u}} = \overline{\mathbf{u}_Y}$.*
(4) *For $\emptyset \subset Y \subset X$, $(D_Y, \mathbf{u}_Y)$ is an exclusive pair.*

Remark. In any graph, a vector supported by a path partition satisfies (B3). Theorem J(3) gives the converse for $J(n,d)$. Thus, $Y \Leftrightarrow D_Y \Leftrightarrow \pi_Y \Leftrightarrow \overline{\mathbf{u}_Y}$ are natural bijections between the nonempty proper subsets of X, the subsets of $\mathscr{V}$ which support a vector in V_1, the path partitions which support such a vector and (up to positive scalar multiple) the vectors $\mathbf{u} \in V_1$ supported by some path partition. Related comments apply to the next two theorems.

The Hamming Graph $H(d,n)$ has as vertices the n^d d-tuples drawn from an n-set X and as edges all pairs of vertices which differ in exactly one coordinate. If $Y_1, \dots, Y_d$ are nonempty subsets of X then $\prod_{i=1}^d Y_i$ is a set of vertices of which we will call a *box*.

Theorem H. *Let $\mathscr{V} = \prod_{i=1}^d X$ be the vertex set of $H(d,n)$.*

(1) *Each box B strongly supports a vector $\mathbf{u}_B \in V_1$.*

(2) *A box* $B=\prod_{i=1}^{d} Y_i$ *is completely regular if and only if there is an integer* q, $1\leqslant q<n$ *such that each* Y_i *either has cardinality* q *or equals* X.
(3) *Suppose that* $\mathbf{u}\in V_1$ *satisfies* (B3) *and* (B4). *Then the set on which it takes its maximum is a box* B *and* $\bar{\mathbf{u}}=\overline{\mathbf{u}_B}$.
(4) *For a box* B, $(B,\mathbf{u}_B)$ *is an exclusive pair.*

The Complete Multipartite Graph $K_{d\times n}$ has dn vertices which fall into d disjoint *blocks* X_i, each of size n. The edges are all pairs of vertices from distinct blocks. Call a set Q of vertices *uniform* if, for some $1\leqslant q<n$, it contains q vertices from each block. Call a set P of vertices *properly independent* if it is a proper subset of some block X_i. Call an ordered pair (P,P'') of nonempty vertex sets an *independent pair* if $P\subset X_i$ is properly independent and $P''\subseteq P'=X_i\backslash P$. For such a pair let $\pi_{PP''}$ denote the path partition $[P,\mathscr{V}-(P\cup P''),P'']$. Thus $\pi_P=\pi_{PP'}$. Call a vector $\mathbf{u}\in\mathbb{R}^{\mathscr{V}}$ *concentrated* if there is a block X_i so that all the nonzero entries of $\mathbf{u}$ are indexed by vertices from X_i.

Theorem K. *Let* $\mathscr{V}=\bigcup_{i=1}^{d} X_i$ *be the vertex set of* $K_{d\times n}$. *For* $k>1$, *let* $V_{1,k}$ *denote the set of* $\mathbf{u}\in V_1$ *whose entries have exactly* k *distinct values.*

(1) (a) *Each uniform set* Q *strongly supports a vector* $\mathbf{u}_Q\in V_{1,2}$.
   (b) *Each properly independent set* P *strongly supports a vector* $\mathbf{u}_P\in V_{1,3}$.
   (c) *For each independent pair* (P,P''), *the path partition* $\pi_{PP''}$ *strongly supports a vector* $\mathbf{u}_{PP''}\in V_{1,3}$. *If* $P''=P'$, *then* $\pi_{PP''}=\pi_P$ *and* $\mathbf{u}_{PP''}=\mathbf{u}_P$. *If* $P''\subset P'$, *then* $\pi_{PP''}$ *is a path partition but not a distance partition.*
(2) *The uniform sets and properly independent sets are completely regular.*
(3) (a) *If* $\mathbf{u}\in V_{1,2}$, *then* $\mathbf{u}$ *satisfies* (B3) *and* (B4) *and* $\bar{\mathbf{u}}=\bar{\mathbf{u}}_Q$ *for the uniform set* $Q=\{\gamma\mid \mathbf{u}(\gamma)>0\}$.
   (b) *If* $\mathbf{u}\in V_{1,3}$, *then* $\mathbf{u}$ *satisfies* (B3) *and* (B4) *if and only if* $\mathbf{u}$ *is concentrated. Then* $\bar{\mathbf{u}}=\bar{\mathbf{u}}_{P,P''}$ *for the independent pair* $(P,P'')=(\{\gamma\mid\mathbf{u}(\gamma)>0\},\{\gamma\mid\mathbf{u}(\gamma)<0\})$.
   (c) *If* $\mathbf{u}\in V_{1,k}$ *for* $k\geqslant 4$, *then no path partition supports* $\mathbf{u}$. *Furthermore,* $\mathbf{u}$ *satisfies* (B3) *and* (B4) *if and only if* $\mathbf{u}$ *is concentrated.*
(4) *Each* $(Q,\mathbf{u}_Q),(P,\mathbf{u}_P)$, *and* $(\pi_{PP''},\mathbf{u}_{PP''})$ *is an exclusive pair.*

Putting parts of the three main theorems together yields Table 1 where $\surd$ denotes that a property always holds and the other symbols are explained below.

3. Preliminary results

We start with an important pair of results about the cycle balance conditions. The remainder of the section sketches some standard results about eigenvalues and distance-regular graphs. The reader may wish to skim it for notation on a first reading, referring back to the proofs as needed. A comprehensive treatment of distance-regular graphs can be found in [1] or [3].

Table 1

	$J(n,d)$	$H(2,n)$	$H(d,n)$ $d>2$	$K_{d\times 2}$	$K_{d\times 3}$	$K_{d\times n}$ $n>3$
I	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\clubsuit$
II	$\surd$	$\surd$	$\surd$	$\surd$	$\diamondsuit$	$\diamondsuit$
III	$\surd$	$\surd$	$\heartsuit$	$\surd$	$\surd$	$\surd$

Note: *I*: If $\mathbf{u}\in V_1$ satisfies (B3) and (B4) then $\mathbf{u}$ is supported by a path partition. ($\clubsuit$) The exceptions are concentrated vectors $\mathbf{u}\in V_{1,k}$ for $k>3$. *II*: If π is a path partition which supports $\mathbf{u}\in V_1$ then π is a distance partition. ($\diamondsuit$) The exceptions are the three-cell partitions $\pi=\pi_{PP''}$ with $P\cup P''$ a proper subset of some block. *III*: If π is a distance partition which supports $\mathbf{u}\in V_1$ then π is an equitable distance partition. Equivalently, If D is a set which supports $\mathbf{u}$ then D is completely regular. ($\heartsuit$) See Theorem H(2).

Lemma 3.1. *Given* $\mathbf{u}\in\mathbb{R}^{\mathscr{V}}$, *say that a cycle* $\alpha,\beta,\gamma\ldots$ *has type* $(a,b,c,\ldots)$ *if these are the values of* $\mathbf{u}(\alpha),\mathbf{u}(\beta),\mathbf{u}(\gamma),\ldots$ *for the vertices taken in some consecutive order. Then a* 3-*cycle is balanced by* $\mathbf{u}$ *if and only if it has type* (a,a,b) *for some numbers* a *and* b *which may be equal. Similarly, a* 4-*cycle is balanced by* $\mathbf{u}$ *if and only if it has one of the types* (a,a,b,b) *or* (a,b,a,c).

Proof. A cycle of any of these types is balanced by $\mathbf{u}$. The only other possible types are $(a,b,c),(a,a,b,c)$ and (a,b,c,d) where a,b,c and d are pairwise unequal. None of these are balanced. $\square$

Remark. If $\mathbf{u}$ takes on only two values, then any cycle is balanced by $\mathbf{u}$.

Given a partition π of $\mathscr{V}(\Gamma)$, let Γ_π denote the simple graph whose vertices are the cells of π with two cells adjacent exactly if there is an edge of Γ going from one to the other. If π is a distance partition then Γ_π is a path. The converse need not be true (see Lemma 6.3(5) or Theorem K 1(c) for an example).

Lemma 3.2. *Let* $\mathbf{u}\in\mathbb{R}^{\mathscr{V}}$ *be supported by a partition* π *of* $\mathscr{V}(\Gamma)$. *If* Γ_π *is a tree, then every oriented cycle in* Γ *is balanced with respect to* $\mathbf{u}$. *In particular, any* $\mathbf{u}$ *supported by a path partition satisfies* (B3) *and* (B4).

Proof. Augment Γ_π by adding a loop at each of its vertices. Then an oriented cycle in Γ corresponds naturally to a closed oriented walk in this augmented Γ_π. Let C and C' be two cells of π. Since Γ_π is a tree, an oriented cycle in Γ must have as many edges going from C to C' as from C' to C. $\square$

Let Γ be a regular simple connected graph with vertex set $\mathscr{V}$ and diameter d. Its *adjacency matrix* A has at least $d+1$ eigenvalues $\theta_0>\theta_1>\cdots$. We also refer to θ_i as an eigenvalue of Γ and let V_i denote the corresponding eigenspace of $\mathbb{R}$. The largest eigenvalue θ_0 is the common degree and V_0 is the span of the all ones vector, $\mathbf{1}$.

Corresponding to an equitable partition $\pi=[C_0,C_1,\ldots,C_\rho]$ of $\mathscr{V}$ is the $(\rho+1)\times(\rho+1)$ *quotient matrix* A/π whose (i,j) entry is s_{ij}. If π is an equitable path partition then this matrix is tridiagonal; $s_{ij}=0$ if $|i-j|>\mathbf{1}$.

Lemma 3.3. *Let $\pi = [C_0, C_1, \ldots, C_\rho]$ be a path partition of $\mathscr{V}$, the vertex set of a simple connected graph Γ. And let $\mathbf{u} \in V_i$ be an eigenvector of Γ supported by π. Then*

(1) $\{\mathbf{w} \in V_i \mid \pi \text{ supports } \mathbf{w}\} = \{\mathbf{w} \in V_i \mid \pi^* \text{ supports } \mathbf{w}\} = \langle \mathbf{u} \rangle$,
(2) *for* $\gamma \in C_0$ *and for* $\gamma \in C_\rho$, $\mathbf{u}(\gamma) \neq 0$.
Suppose that π *is also equitable.*
(3) A/π *has* $\rho + 1$ *distinct eigenvalues, each of which is an eigenvalue of* A.
(4) *A number* θ *is an eigenvalue of* A/π *if and only if* π *supports an eigenvector of* A *for* θ.
(5) *The largest eigenvalue of* A, *namely the common degree, is also an eigenvalue of* A/π.

Proof. The first two sets in (1) are clearly equal and contain $\langle \mathbf{u} \rangle$. Let $\gamma_0, \ldots, \gamma_\rho$ be vertices with $\gamma_i \in C_i$ adjacent to some vertex in C_{i+1} for $i < \rho$ and let $\hat{A}$ be the tridiagonal matrix whose (i,j) entry is the number of vertices of C_j adjacent to γ_i. Consider the natural bijection from $\mathbb{R}^{\rho+1}$ to $\{\mathbf{w} \in \mathbb{R}^{\mathscr{V}} \mid \pi \text{ supports } \mathbf{w}\}$ which sends $\mathbf{v} = (v_0, \ldots, v_\rho)^t$ to the vector $\mathbf{w}$ defined by $\mathbf{w}(\gamma) = v_j$ for $\gamma \in C_j$. If this $\mathbf{w}$ is an eigenvector of A for some eigenvalue θ_i, then $\mathbf{v}$ is easily seen be an eigenvector of $\hat{A}$ for the same eigenvalue. From the form of $\hat{A}$ it is clear that θ_i and v_0 uniquely determine $\mathbf{v}$. This proves (1) since $\mathbf{v}$ uniquely determines $\mathbf{w}$. Taking $\mathbf{w} = \mathbf{u}$ we see that θ_i and $\mathbf{u}(\gamma_0)$ uniquely determine $\mathbf{u}$. This proves (2) for C_0, since $\mathbf{u}$ is nonzero. Replace π by its reverse π^* to get (2) for C_ρ.

The form of $\hat{A}$ shows that its characteristic polynomial has degree $\rho + 1$. The proof of (1) shows that every eigenspace of $\hat{A}$ has dimension one. This gives the first part of (3) since $A/\pi = \hat{A}$ when π is equitable. The rest of (3) follows from (4), which we now prove. Suppose a column vector $\mathbf{v} \in \mathbb{R}^{\rho+1}$ and a vector $\mathbf{w} \in \mathbb{R}^{\mathscr{V}}$ supported by π are related by the bijection above. We already noted that $\mathbf{v}$ is an eigenvector of A/π for θ when $\mathbf{w}$ is an eigenvector of A for the same eigenvalue. When π is equitable, the converse implication holds. Finally, (5) follows from (4) since any partition supports $\mathbf{1}$. $\square$

Remark. It would be interesting to know if (5) is true when π is an equitable partition of a graph which is not regular.

Lemma 3.4. *Let $\mathbf{y} \in \mathbb{R}^{\mathscr{V}}$ be a vector supported by an equitable two cell partition π. If the trace of the 2×2 quotient matrix A/π is $\theta_0 + \theta_1$ then $\mathbf{y} \in V_0 \oplus V_1$.*

Proof. The trace of A/π is the sum of its two eigenvalues. By Lemma 3.3(5), one is θ_0 so the other is θ_1. By Lemma 3.3(4) π supports some $\mathbf{u} \in V_1$. Clearly $\mathbf{y}$ is a linear combination of $\mathbf{1} \in V_0$ and $\mathbf{u}$. $\square$

When Γ is a distance regular graph, as it is in all the main theorems, the distance partition with respect to a single vertex is equitable. Then Γ has exactly $d + 1$ eigenvalues and the path partition with respect to distance from a given vertex supports an eigenvector, unique up to scalar multiple, for each. We refer to a set of $|\mathscr{V}|$ eigenvectors for θ_i, one supported by each vertex, as a *standard spanning set* for V_i.

Remark. The idempotent matrix E_i giving the orthogonal projection from $\mathbb{R}^{\mathscr{V}}$ into V_i is a polynomial in A. Its rows constitute a standard spanning set. We will scale up to get a set of integer vectors.

Let $\pi=[C_0,\dots,C_\rho]$ be a path partition. When $|i-j|=1$, at least one $\gamma\in C_i$ has a neighbor in C_j. When π is an equitable path partition, all $\gamma\in C_i$ have s_{ij} such neighbors and hence π satisfies

$$\text{If } |i-j|=1 \text{ then every vertex in } C_i \text{ is adjacent to a vertex in } C_j. \tag{X}$$

We need some consequences of this rather weak condition. Recall that Γ is assumed to be connected and regular.

Lemma 3.5. *Let $\pi=[C_0,\dots,C_\rho]$ be a path partition.*

(1) *If π satisfies (X), and in particular if π is equitable, then π is the distance partition π_{C_0}.*
(2) *Suppose π satisfies (X) and strongly supports a vector $\mathbf{u}\in\mathbf{V}_1$. Then the following condition is sufficient to show that $(\pi,\mathbf{u})$ is an exclusive pair:*
   - *The induced graph on C_0 is connected as is the induced graph on C_ρ*

Proof. If π is a path partition and $\gamma\in C_i$, then $\mathrm{dist}(\gamma,C_0)\geqslant i$. Under condition (X), $\mathrm{dist}(\gamma,C_0)\leqslant i$. *Hence* $\pi=\pi_{C_0}$.

Under the assumptions of (2), π strongly supports $\mathbf{u}$ and π^* supports $\mathbf{u}$. This gives the first and part of the last of the three conditions for $(\pi,\mathbf{u})$ to be an exclusive pair. Lemma 3.3(1) shows that the second condition is automatically true. What remains to be shown is uniqueness. The maximum value of $\mathbf{u}$ is taken on C_0 and the minimum on C_ρ. Since $\mathbf{u}$ is orthogonal to 1, the first is positive and the second negative. Let $\pi'=[C_0',C_1',\dots]$ be a path partition which supports $\mathbf{u}$. From Lemma 3.3(2), the value taken by $\mathbf{u}$ on C_0' cannot be zero. We will assume it is positive and show that $\pi'=\pi$. Note that each cell of π' is contained in some cell of π since $\mathbf{u}(\gamma)$ is constant on each cell of π' and different on different cells of π. Hence, π' has at least $\rho+1$ cells. We first show that $C_0'\subseteq C_0$ and $C_1'\subseteq C_1$. Fix a vertex $\gamma\in C_0'$. Since $\mathbf{u}(\gamma)$ is positive, $\gamma\notin C_\rho$. Since the entries $\mathbf{u}(\delta)$ for δ adjacent to γ take exactly one value other than $u(\gamma)$, $\gamma\notin C_i$ for any $0<i<\rho$. Hence, $\gamma\in C_0$ and $C_0'\subseteq C_0$. Choose a path $\gamma_0,\gamma_1\dots,\gamma_\rho$ with $\gamma=\gamma_0$ and $\gamma_i\in C_i$. $C_1'\subseteq C_1$ since $\gamma_1\in C_1$ is adjacent to $\gamma_0\in C_0'$ but $\gamma_1\notin C_0'$.

We now show that $C_0'=C_0$. Let γ_0 be as above and δ be any other vertex in C_0. Since the induced graph on C_0 is connected, there is a path from γ_0 to δ which includes no vertices of C_1. This path certainly includes no vertices of C_1' and hence lies entirely in C_0'. Thus $\delta\in C_0'$ and $C_0'=C_0$ as desired.

This suffices to prove $\pi=\pi'$ since a path $\delta_0,\dots,\delta_\rho$ with $\delta_i\in C_i$ for each i has $\delta_0\in C_0'$ so $\delta_i\in C_i'$ for each i. Since δ_i can be any vertex of C_i, $C_i'=C_i$ and the result follows. $\square$

Remark. We noted that there are path partitions $\pi_{PP''}$ in $K_{d\times n}$ which are not distance partitions. They do not satisfy condition (X). We will still be able to show that they belong to exclusive pairs.

4. The case $J(n, d)$

The Johnson Graph $J(n,d)$ has as vertices the d-sets of an n-set X, $n>d>0$. A pair (γ,δ) is an edge if $\gamma\cap\delta$ has cardinality $d-1$. Hence, $dist(\gamma,\delta)=d-|\gamma\cap\delta|$. The two largest eigenvalues are $\theta_0=d(n-d)$ and $\theta_1=(d-1)(n-d-1)-1$ [1, 9.1]. We will assume that $X=\{1,2,\ldots,n\}$. Because $J(n,d)$ and $J(n,n-d)$ are isomorphic, it is usual to require $2d\leqslant n$. We will not need this restriction.

For $1\leqslant i\leqslant n$ let $\mathbf{x}_i\in\mathbb{R}^{\mathscr{V}}$ be the vector (of length $\binom{n}{d}$) defined by

$$\mathbf{x}_i(\gamma)=\begin{cases}1 & i\in\gamma,\\ 0 & i\notin\gamma.\end{cases}$$

Lemma 4.1. *With the definitions above,*

(1) *The n vectors $\mathbf{x}_i$ span a subspace $W\subseteq V_0\oplus V_1$.*
(2) *$W\cap V_1$ consists of all vectors which can be expressed in the form*

$$\mathbf{u}=\sum_{i=1}^{n}\alpha_i\mathbf{x}_i \quad \textit{and} \quad \sum_{i=1}^{n}\alpha_i=0. \tag{J}$$

Proof. We will see in Lemma 4.3 that the containment in (1) is actually equality. The weaker statement here follows from Lemma 3.4 since the two cell partition supporting $\mathbf{x}_i$ is equitable and the trace of its quotient matrix is $(d-1)(n-d)+d(n-d-1)=\theta_0+\theta_1$. Then (2) follows since it describes the $\mathbf{u}\in W$ which are orthogonal to the all ones vector $\mathbf{1}$ and hence in V_1. $\square$

For $\emptyset\subset Y\subset X$, define the *shadow* of Y to be the set

$$D_Y=\{\gamma\subset X\mid |\gamma|=d \text{ and either } \gamma\subseteq Y \text{ or } Y\subseteq\gamma\}\subset\mathscr{V},$$

π_Y to be the distance partition of $\mathscr{V}$ with respect to D_Y, and let

$$\mathbf{u}_Y=\sum_{i\in Y}(n-q)\mathbf{x}_i-\sum_{i\notin Y}q\mathbf{x}_i \quad \text{where } q=|Y|.$$

Lemma 4.2. *For $\emptyset\subset Y\subset X$ let $Y'=X\setminus Y$.*

(1) $\mathbf{u}_{Y'}=-\mathbf{u}_Y$.
(2) $\mathbf{u}_Y\in V_1$.
(3) *Let $q=|Y|$ and $m=\min(d,q)$. Then π_Y has $m+1$ cells and $\mathbf{u}_Y(\gamma)=mn-q^2-kn$ for $\gamma\in C_k$.*
(4) *π_Y strongly supports $\mathbf{u}_Y$. Furthermore,*

(5) $(D_Y, \mathbf{u}_Y)$ *is an exclusive pair.*
(6) D_Y *is a completely regular subset of* $\mathscr{V}$.

Proof. (1) is immediate from the definitions. The sums defining $\mathbf{u}_Y$ have the form (J), so (2) follows from Lemma 4.1(2). The first part of (3) is clear, the cell C_k consists of all vertices γ with $|\gamma \cap Y| = m - k$. Each entry $\mathbf{u}_Y(\gamma)$ is a sum of $|Y| = q$ numbers. For $\gamma \in C_k$, $m - k$ of these numbers are $n - q$ and the other $q - (m - k)$ are $-q$. This gives the rest of (3) and (4). Since the induced graphs on D_Y and on $D_{Y'}$ are connected, (5) follows from Lemma 3.5(2). It remains only to show (6). For this, we must show that the constants s_{ij} are well defined. A vertex has $|X \backslash \gamma| \cdot |\gamma| = (n - d)d$ neighbors. For $\gamma \in C_k$, $|Y \backslash \gamma| \cdot |\gamma \backslash Y| = (q - (m - k)) \cdot (d - (m - k))$ of them are in C_{k-1} and $|X \backslash (\gamma \cup Y)| \cdot |\gamma \cap Y| = (n - (d + q - (m - k))) \cdot (m - k)$ are in C_{k+1}. The remainder are in C_k. $\square$

Remark. Any one of $Y, D_Y, \mathbf{u}_Y$ and π_Y uniquely determines the rest.

Lemma 4.3. *With the definitions above,*

(1) *The span* W *of the* n *vectors* $\mathbf{x}_i$ *is* $V_0 \oplus V_1$.
(2) *Suppose* $\mathbf{u} \in V_1$ *satisfies* (B3). *Then the set on which it takes its maximum is a shadow* D_Y *and* $\bar{\mathbf{u}} = \overline{\mathbf{u}_Y}$.

Proof. We already know that $V_0 \subset W \subseteq V_0 \oplus V_1$. Each of the $\binom{n}{d}$ vertices γ is a d element subset of X. Their shadows are the singleton sets $\{\gamma\}$ and (1) follows since the corresponding vectors $\mathbf{u}_{\{\gamma\}} \in W$ make up a standard spanning set for V_1. Thus, any $\mathbf{u} \in V_1$ can be written in the form (J) as $\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$ with $\sum_{i=1}^{n} \alpha_i = 0$. We will show that, when $\mathbf{u}$ satisfies (B3), the α_i take two distinct values, one positive and the other negative. Then (2) follows since, up to positive scalar multiple, the only vector satisfying all these conditions is $\mathbf{u}_Y = \sum_{i \in Y} (n - q)\mathbf{x}_i + \sum_{i \notin Y} (-q)\mathbf{x}_i$ where $Y = \{i \mid \alpha_i > 0\}$ and $|Y| = q$.

The n constants α_i take at least two values. It remains only to show that they do not take more. Assume first that $d \neq n - 1$, consider any three of the constants, say $\alpha_1, \alpha_2, \alpha_3$, and choose a $d - 1$ set γ' disjoint from $\{1, 2, 3\}$. Then the three vertices $\{j\} \cup \gamma'$ of $J(n, d) j = 1, 2, 3$ are pairwise adjacent. Because $\mathbf{u}$ satisfies (B3), the three entries $\mathbf{u}(\{j\} \cup \gamma')$ can take at most two distinct values. But $\mathbf{u}(\{j\} \cup \gamma') = \alpha_j + \sum_{i \in \gamma'} \alpha_i$, so the three constants α_1, α_2, and α_3 take at most two distinct values.

In the case $d = n - 1$ (which is a complete graph), the three vertices $X \backslash \{j\}$, $j = 1, 2, 3$ form a triangle and the corresponding entries of $\mathbf{u}$ are $-\alpha_j$. Again these can take at most two distinct values. $\square$

Lemmas 4.2(4), 4.2(6),4.3(2) and 4.2(5) give the four claims of

Theorem J. *Let* $\mathscr{V} = \{\gamma \subset X \mid |\gamma| = d\}$ *be the vertex set of* $J(n, d)$.

(1) *For each* $\emptyset \subset Y \subset X$, *the shadow* D_Y *strongly supports a vector* $\mathbf{u}_Y \in V_1$.
(2) *Each* D_Y *is completely regular.*

(3) *Suppose* $\mathbf{u} \in V_1$ *satisfies* (B3). *Then the set on which it takes its maximum is a shadow* D_Y *and* $\bar{\mathbf{u}} = \overline{\mathbf{u}_Y}$.
(4) *For* $\emptyset \subset Y \subset X$, $(D_Y, \mathbf{u}_Y)$ *is an exclusive pair.* □

5. The case $H(d, n)$

The Hamming Graph $H(d,n)$ has as vertices the n^d d-tuples drawn from an n-set X, $n > 1$ and $d \geqslant 1$, so a typical element is $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d)$. The edges are all pairs of vertices which differ in exactly one coordinate. Hence $\mathrm{dist}(\gamma, \delta) = |\{i \mid \gamma_i \neq \delta_i\}|$. The two largest eigenvalues are $\theta_0 = dn - d$ and $\theta_1 = dn - d - n$. [1, 9.2]. We will assume that $X = \{1, 2, \ldots, n\}$.

For $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant n$ let $\mathbf{z}_{ij}$ be the vector

$$\mathbf{z}_{ij}(\gamma) = \begin{cases} 1 & \gamma_i = j, \\ 0 & \gamma_i \neq j. \end{cases}$$

Lemma 5.1. *With the definitions above,*

(1) *The nd vectors* $\mathbf{z}_{ij}$ *span a subspace* $W \subseteq V_0 \oplus V_1$.
(2) $W \cap V_1$ *consists of all vectors which can be expressed in the form*

$$\mathbf{u} = \sum \alpha_{ij} \mathbf{z}_{ij} \quad and \quad \sum_{j=1}^{n} \alpha_{ij} = 0 \quad for\ 1 \leqslant i \leqslant d. \tag{H}$$

(3) *If* $\mathbf{u} = \sum \alpha_{ij} \mathbf{z}_{ij}$, *then each entry is a sum of d coefficients*; $\mathbf{u}(\gamma) = \sum \alpha_{i\gamma_i}$.

Proof. We will see in Lemma 5.3 that the containment in (1) is actually equality. The weaker statement here follows from Lemma 3.4 since the two cell partition supporting $\mathbf{z}_{ij}$ is equitable and the trace of its quotient matrix is $(d-1)(n-1) + ((d-1)(n-1) + (n-2)) = \theta_0 + \theta_1$.

Any vector expressed in the form (H) belongs to the set of vectors which can be expressed in the form

$$\mathbf{u} = \sum \alpha_{ij} \mathbf{z}_{ij} \quad \text{and} \quad \sum_{i=1}^{d} \sum_{j=1}^{n} \alpha_{ij} = 0.$$

These sums are precisely those vectors of W orthogonal to the all ones vector $\mathbf{1}$ and hence in V_1. It remains to show that any vector expressed in this form can also be expressed in the more restrictive form (H). For each fixed i, let $\overline{\alpha_i} = \sum_{j=1}^{n} \alpha_{ij}/n$ and define $\alpha'_{ij} = \alpha_{ij} - \overline{\alpha_i}$. Observe that $\sum_{j=1}^{n} \mathbf{z}_{ij} = \mathbf{1}$. Hence $\sum \alpha'_{ij} \mathbf{z}_{ij} = \sum \alpha_{ij} \mathbf{z}_{ij} = \mathbf{u}$ since the difference is

$$\sum_{i=1}^{d} \sum_{j=1}^{n} \overline{\alpha_i} \mathbf{z}_{ij} = \sum_{i=1}^{d} (\overline{\alpha_i} \mathbf{1}) = \frac{1}{n} \left(\sum_{i=1}^{d} \sum_{j=1}^{n} \alpha_{ij} \right) \mathbf{1}$$

and the value of the double sum on the right is zero. So, as desired, $\mathbf{u} = \sum \alpha'_{ij}\mathbf{z}_{ij}$ has the form (H). This establishes (2). The definition of the nd vectors z_{ij} gives (3). □

If $Y_1, \dots, Y_d$ are nonempty subsets of X, then $B = \prod_{i=1}^{d} Y_i$ is a set of vertices of which we will call a *box*. For a box B, let $q_i = |Y_i|$ and

$$\mathbf{u}_B = \sum \alpha_{ij}\mathbf{z}_{ij} \quad \text{where } \alpha_{ij} = \begin{cases} n - q_i & j \in Y_i, \\ -q_i & j \notin Y_i. \end{cases}$$

In addition, let B' be the box $\prod_{i=1}^{d} Y'_i$ with

$$Y'_i = \begin{cases} X \backslash Y_i & Y_i \neq X, \\ X & Y_i = X. \end{cases}$$

Lemma 5.2. *With the definitions above,*

(1) $\mathbf{u}_{B'} = -\mathbf{u}_B$.
(2) $\mathbf{u}_B \in V_1$.
(3) *Let* $q_i = |Y_i|$ *and* $m = |\{i \mid q_i < q\}|$ *be the number of* Y_i *which are proper subsets of* X. *Then* π_B *has* $m+1$ *cells and* $\mathbf{u}_B(\gamma) = dn - \sum_{i=1}^{d} q_i - kn$ *for* $\gamma \in C_k$.
(4) π_B *strongly supports* $\mathbf{u}_B$.
(5) $(B, \mathbf{u}_B)$ *is an exclusive pair.*
(6) B *is completely regular if and only if there is an integer* q, $1 \leqslant q < n$, *such that each* Y_i *either has cardinality* q *or equals* X.

Proof. (1) follows from the definitions. The sums defining $\mathbf{u}_B$ have the form (H) so (2) follows from 5.1(2).

The cell C_k of π_B, $0 \leqslant k \leqslant m$, consists of all vertices γ so that $|\{i \mid \gamma_i \notin Y_i\}| = k$. The entry $\mathbf{u}_B(\gamma)$ is a sum of d numbers,

$$\mathbf{u}_B(\gamma) = \sum_{\{i \mid \gamma_i \in Y_i\}} (n - q_i) + \sum_{\{i \mid \gamma_i \notin Y_i\}} (-q_i).$$

For $\gamma \in C_k$, this is $(d-k)n - \sum_{i=1}^{d} q_i$. This gives (3) and (4). Since the induced graphs on B and on B' are connected, (5) follows from Lemma 3.5(2). It remains to prove (6). Recall that s_{ij} is the number of vertices of C_j adjacent to any given vertex in C_i. We must determine when these constants are well defined. Fix a vertex $\gamma \in C_k$ and let $S = \{i \mid \gamma_i \notin Y_i\}$. Then $|S| = k$ and $S \subseteq T = \{i \mid Y_i \neq X\}$. Of the $d(n-1)$ neighbors of γ, $\sum_{i \in S} q_i$ are in C_{k-1} and $\sum_{i \in T \backslash S} (n - q_i)$ are in C_{k+1}. The remainder are in C_k. B is completely regular precisely when these three quantities depend only on k and not on the particular choice of $\gamma \in C_k$. Since S can be any k element subset of T, this occurs exactly when there is an integer q, $1 \leqslant q < n$, such that $q_i = q$ for all $i \in T$. □

Lemma 5.3. *With the definitions above,*

(1) *The span of the* nd *vectors* z_{ij} *is* $W = V_0 \oplus V_1$.

(2) *Suppose that* $\mathbf{u} \in V_1$ *satisfies* (B3) *and* (B4). *Then the set on which it takes its maximum is a box* B *and* $\bar{\mathbf{u}} = \overline{\mathbf{u}_B}$.

Proof. We already know that $V_0 \subset W \subseteq V_0 \oplus V_1$. Each of the n^d singleton vertex sets $\{\gamma\}$ is a box. Since the corresponding vectors $\mathbf{u}_{\{\gamma\}} \in W$ make up a standard spanning set for V_1, (1) follows.

Suppose that $\mathbf{u} \in V_1$ satisfies (B3) and (B4). Write $\mathbf{u}$ in the form (H) and consider the box $B = \prod Y_i$ where $Y_i = \{j \mid \alpha_{ij} \geqslant 0\}$. We first use (B3) to show that for each fixed i, the n coefficients α_{ij} take at most two distinct values.

Consider the d vertices with all coordinates except the ith equal to 1. If γ has this form with $\gamma_i = j \geqslant 1$, then $\mathbf{u}(\gamma) = \alpha_{ij} + \sum_{k \neq i} \alpha_{k1}$. Because any three of these vertices are adjacent, the d entries $\mathbf{u}(\gamma)$ take at most two distinct values. Hence, the same is true of the d coefficients α_{ij}. Let $q_i = |Y_i|$. Since these d coefficients sum to zero, either $Y_i = X$ and all are zero or else there is a constant $\lambda_i > 0$ such that

$$\alpha_{ij} = \begin{cases} \lambda_i(n - q_i) & j \in Y_i, \\ -\lambda_i q_i & j \notin Y_i. \end{cases}$$

We now use (B4) to show that all the defined λ_i are equal to a common value λ and thus that $\mathbf{u} = \lambda \mathbf{u}_B$. This will give (2).

Choose two values of i such that $Y_i \neq X$, say $i = 1$ and $i = 2$. Then $\lambda_1 > 0$ and $\lambda_2 > 0$. We wish to show that $\lambda_1 = \lambda_2$. There is no loss of generality in assuming that $\alpha_{i1} = \lambda_i(n - q_i)$ and $\alpha_{in} = -\lambda_i q_i$ for $i = 1, 2$. Let γ^{ab} denote the vertex $(a, b, 1, 1, \ldots)$ and consider the four vertices $\gamma^{11}, \gamma^{1n}, \gamma^{nn}, \gamma^{n1}$. They constitute a 4-cycle. Since $\mathbf{u}(\gamma^{ab}) = \alpha_{1a} + \alpha_{2b} + \sum_3^d \alpha_{i1}$, $\mathbf{u}(\gamma^{11}) > \mathbf{u}(\gamma^{1n}) > \mathbf{u}(\gamma^{nn})$ and $\mathbf{u}(\gamma^{11}) > \mathbf{u}(\gamma^{n1}) > \mathbf{u}(\gamma^{nn})$. The only way that this 4-cycle can be balanced is if $\mathbf{u}(\gamma^{1n}) = \mathbf{u}(\gamma^{n1})$. This means that $\lambda_1(n - q_1) - \lambda_2 q_2 = -\lambda_1 q_1 + \lambda_2(n - q_2)$ and hence $n(\lambda_1 - \lambda_2) = 0$ as desired. Since $\lambda > 0$ and $\mathbf{u} = \lambda \mathbf{u}_B$, $\bar{\mathbf{u}} = \overline{\mathbf{u}_B}$. $\square$

Note that both $\mathbf{z}_{11} - \mathbf{z}_{12}$ and $\mathbf{z}_{11} + \mathbf{z}_{12} - 2\mathbf{z}_{21}$ are in V_1. For $n > 2$, the first satisfies (B4) but not (B3) and the other satisfies (B3) but not (B4).

Lemmas 5.2(4), 5.2(6), 5.3(2) and 5.2(5) give the three claims of

Theorem H. *Let* $\mathscr{V} = \prod_{i=1}^d X$ *be the vertex set of* $H(d, n)$.

(1) *Each box* B *strongly supports a vector* $\mathbf{u}_B \in V_1$.
(2) *A box* $B = \prod_{i=1}^d Y_i$ *is completely regular if and only if there is an integer* q, $1 \leqslant q < n$ *such that each* Y_i *either has cardinality* q *or equals* X.
(3) *Suppose that* $\mathbf{u} \in V_1$ *satisfies* (B3) *and* (B4). *Then the set on which it takes its maximum is a box* B *and* $\bar{\mathbf{u}} = \overline{\mathbf{u}_B}$.
(4) *For a box* B, $(B, \mathbf{u}_B)$ *is an exclusive pair.*

6. The case $K_{d \times n}$

The Complete Multipartite Graph $K_{d \times n}$ has dn vertices which fall into d disjoint *blocks* X_i, each of size n. The edges are all pairs of vertices from distinct blocks. We will assume $d>1$ and $n>1$. The smallest of the three eigenvalues is negative and the other two are $\theta_0=(d-1)n$ and $\theta_1=0$.

Lemma 6.1. *V_1 consists of all vectors indexed by $\mathscr{V}$ such that*

$$\sum_{\gamma\in X_i}\mathbf{u}(\gamma)=0 \quad \textit{for } 1\leqslant i\leqslant d. \tag{K}$$

Proof. In any regular graph, if $\mathbf{u}\in V_1$ then $\sum_{\gamma\in V}\mathbf{u}(\gamma)=0$. Since $\theta_1=0$, V_1 consists of all vectors $\mathbf{u}\in\mathbb{R}^{\mathscr{V}}$ such that, for each fixed i, $\sum_{\gamma\notin X_i}\mathbf{u}(\gamma)=0$. The result follows. □

Remark. For $1\leqslant i\leqslant d$, let $\mathbf{z}_{ij}$ $1\leqslant j\leqslant n$ be the standard basis vectors corresponding to the vertices in block X_i. Then V_1 consists of all vectors which can be expressed in the form

$$\mathbf{u}=\sum\alpha_{ij}\mathbf{z}_{ij} \quad\text{and}\quad \sum_{j=1}^{n}\alpha_{ij}=0 \quad \text{for } 1\leqslant i\leqslant d.$$

This description is formally identical to that for $H(d,n)$. Here, however, the symbols $\mathbf{z}_{ij}$ stand for the standard basis of $\mathbb{R}^{dn}$ rather than a dependent set of size dn in $\mathbb{R}^{n^d}$ which spans $V_0\oplus V_1$ for $H(d,n)$.

Call a vertex set P *properly independent* if it is a nonempty proper subset of one of the blocks X_i. For such a P, let $q=|P|, P'=X_i\backslash P$ and $\mathbf{u}_P$ be the vector defined by

$$\mathbf{u}_P(\gamma)=\begin{cases} n-q & \gamma\in P,\\ -q & \gamma\in P',\\ 0 & \gamma\notin X_i.\end{cases}$$

Call a set Q of qd vertices *uniform* if $q<n$ and $|Q\cap X_i|=q$ for each i. For such a Q, let $Q'=\mathscr{V}\backslash Q$ and let $\mathbf{u}_Q$ be the vector defined by

$$\mathbf{u}_Q(\gamma)=\begin{cases} n-q & \gamma\in Q,\\ -q & \gamma\in Q'.\end{cases}$$

Lemma 6.2. *For $D\subset\mathscr{V}$ which is properly independent or uniform*

(1) $\mathbf{u}_{D'}=-\mathbf{u}_D$.
(2) $\mathbf{u}_D\in V_1$.
(3) *π_D strongly supports $\mathbf{u}_D$.*
(4) *$(D,\mathbf{u}_D)$ is an exclusive pair.*
(5) *D is completely regular.*

Proof. Lemma 6.1 and the definitions give everything except (4). For uniform D, argue directly or use Lemma 3.5(2). For D properly independent, note that no path partition supporting $\mathbf{u}_D$ can have less than three cells and no path partition in $K_{d\times n}$ can have more. □

Call an ordered pair (P, P'') of vertex sets an *independent pair* if P is properly independent and $P'' \subseteq P'$. For such a pair let $q = |P|$, $q'' = |P''|$, $\mathbf{u}_{PP''}$ be the vector defined by

$$\mathbf{u}_{PP''}(\gamma) = \begin{cases} q'' & \gamma \in P, \\ -q & \gamma \in P'', \\ 0 & \gamma \notin P \cup P'' \end{cases}$$

and $\pi_{PP''}$ be the ordered partition $[C_0, C_1, C_2] = [P, \mathscr{V} \backslash (P \cup P''), P'']$.

Lemma 6.3. *For an independent pair* (P, P'')

(1) $\mathbf{u}_{PP''} \in V_1$
(2) $\mathbf{u}_{P''P} = -\mathbf{u}_{PP''}$
(3) $\pi_{PP''}$ *strongly supports* $\mathbf{u}_{PP''}$. *Furthermore,*
(4) $(\pi_{PP''}, \mathbf{u}_{PP''})$ *is an exclusive pair.*
(5) *If* $P'' = P'$ *then* $\pi_{PP''} = \pi_P$ *and* $\mathbf{u}_{PP''} = \mathbf{u}_P$. *Otherwise* $\pi_{PP''}$ *is a path partition but not a distance partition.*

Proof. Lemma 6.1 and the definitions above give (1)–(3). There are three conditions for (4). The first has just been shown and the second is Lemma 3.3(1). The third condition follows from the fact that no path partition supporting $\mathbf{u}_{PP''}$ can have less than three cells and no path partition in $K_{d\times n}$ can have more.

The case $P'' = P'$ of (5) is clear. Otherwise, $P \cup P'' \subset X_i$ but X_i has vertices not in $P \cup P''$. These are in the cell C_1 and have all neighbors also in C_1. Hence they are not at distance 1 from $C_0 = P$ (nor from $C_2 = P''$). □

Remark. For $P \cup P'' \subsetneq X_i$ there is an equitable 4-cell partition π which supports $\mathbf{u}_{PP''}$. However, Γ_π is then a star but not a path.

Recall that a vector $\mathbf{u}$ is said to be concentrated when there is a block X_i such that all nonzero entries of $\mathbf{u}$ are indexed by vertices in X_i. Thus, $\mathbf{u}_{PP''}$ is concentrated for an independent pair (P, P''). If $n > 2$, then $K_{d\times n}$ has many other concentrated vectors in V_1. Any concentrated vector whose entries sum to zero satisfies (K) and hence is in V_1. Recall also that $V_{1,k}$ consists of the vectors in V_1 with k distinct entries.

Lemma 6.4. *Suppose that* $\mathbf{u} \in V_1$.

(1) $\mathbf{u}$ *satisfies the cycle-balance conditions* (B3) *and* (B4) *if and only if* $\mathbf{u} \in V_{1,2}$ *or* $\mathbf{u}$ *is concentrated.*
(2) *If* $\mathbf{u} \in V_{1,2}$ *then* $Q = \{\gamma \mid \mathbf{u}(\gamma) > 0\}$ *is a uniform set and* $\bar{\mathbf{u}} = \overline{\mathbf{u}_Q}$.
(3) *If* $\mathbf{u}$ *is concentrated and* $\mathbf{u} \in V_{1,3}$ *then* $\bar{\mathbf{u}} = \overline{\mathbf{u}_{PP''}}$ *for* $P = \{\gamma \mid \mathbf{u}(\gamma) > 0\}$ *and* $P'' = \{\gamma \mid \mathbf{u}(\gamma) < 0\}$.

(4) *If* $\mathbf{u}$ *is concentrated but* $\mathbf{u}\notin V_{1,3}$ *then* $\mathbf{u}\in\mathbf{V}_{1,k}$ *for* $k\geqslant 4$ *and is not supported by a path partition.*

Proof. It follows from Lemma 3.1, or directly, that either possibility of (1) is sufficient for $\mathbf{u}$ to satisfy the cycle-balance conditions. For the other direction, we assume that $\mathbf{u}\in V_1$ satisfies (B3) and (B4) but is not concentrated and show that $\mathbf{u}\in V_{1,2}$. If $\mathbf{u}$ is not concentrated then some two blocks, say X_1 and X_2, contain vertices indexing nonzero entries of $\mathbf{u}$. We will show that the nd entries of $\mathbf{u}$ take exactly two values.

For each block X_i we have $\sum_{\gamma\in X_i}\mathbf{u}(\gamma)=0$. When one of these d entries is nonzero, in particular for $i=1$ and 2, there is another entry of the same block unequal to it. Choose any entries $\mathbf{u}(\gamma_1)>\mathbf{u}(\gamma_1')$ and $\mathbf{u}(\gamma_2)>\mathbf{u}(\gamma_2')$ with $\gamma_1,\gamma_1'\in X_1$ and $\gamma_2,\gamma_2'\in X_2$. These vertices constitute a 4-cycle. Then (B4) forces $\mathbf{u}(\gamma_1)=\mathbf{u}(\gamma_2)>\mathbf{u}(\gamma_1')=\mathbf{u}(\gamma_2')$. Hence, $\mathbf{u}(\gamma)$ has one of these two values for any vertex of X_1 and X_2. Any other vertex γ forms a 3-cycle with γ_1 and γ_2'. So (B3) forces $\mathbf{u}(\gamma)=\mathbf{u}(\gamma_1)$ or $\mathbf{u}(\gamma)=\mathbf{u}(\gamma_2')$. Thus, the nd entries of $\mathbf{u}$ take exactly two values, one positive and the other negative. This establishes (1).

For (2), let $\mathbf{u}$ and Q be as required, $q=|X_1\cap Q|$ and let $\lambda>0$ be the constant such that the negative entries are $-\lambda q$. Since $\mathbf{u}$ satisfies (K), the positive entries are $\lambda(n-q)$ and every block has q vertices of Q. Hence, $\mathbf{u}=\lambda\mathbf{u}_Q$ as desired.

Let $\mathbf{u}\in V_1$ be concentrated. Of the entries $\mathbf{u}(\gamma)$, some are positive, some are zero, and some are negative. $\mathbf{u}\in V_{1,3}$ if and only if all entries of the same sign are equal. This forces $\bar{\mathbf{u}}$ to be $\overline{\mathbf{u}_{PP''}}$. For (4), note that no path partition in $K_{d\times n}$ has at most three cells since $\operatorname{dist}(C_i,C_j)\geqslant|i-j|$. □

Theorem K. *Let* $\mathscr{V}=\bigcup_{i=1}^{d}X_i$ *be the vertex set of* $K_{d\times n}$. *For* $k>1$, *let* $V_{1,k}$ *denote the set of* $\mathbf{u}\in V_1$ *whose entries have exactly* k *distinct values.*

(1) (a) *Each uniform set* Q *strongly supports a vector* $\mathbf{u}_Q\in V_{1,2}$.
   (b) *Each properly independent set* P *strongly supports a vector* $\mathbf{u}_P\in V_{1,3}$.
   (c) *For each independent pair* (P,P''), *the path partition* $\pi_{PP''}$ *strongly supports a vector* $\mathbf{u}_{PP''}\in V_{1,3}$. *If* $P''=P'$, *then* $\pi_{PP''}=\pi_P$ *and* $\mathbf{u}_{PP''}=\mathbf{u}_P$. *If* $P''\subset P'$, *then* $\pi_{PP''}$ *is a path partition but not a distance partition.*

(2) *The uniform sets and properly independent sets are completely regular.*

(3) (a) *If* $\mathbf{u}\in V_{1,2}$, *then* $\mathbf{u}$ *satisfies* (B3) *and* (B4) *and* $\bar{\mathbf{u}}=\bar{\mathbf{u}}_Q$ *for the uniform set* $Q=\{\gamma\,|\,\mathbf{u}(\gamma)>0\}$.
   (b) *If* $\mathbf{u}\in V_{1,3}$, *then* $\mathbf{u}$ *satisfies* (B3) *and* (B4) *if and only if* $\mathbf{u}$ *is concentrated. Then* $\bar{\mathbf{u}}=\overline{\mathbf{u}_{P,P''}}$ *for the independent pair* $(P,P'')=(\{\gamma\,|\,\mathbf{u}(\gamma)>0\},\{\gamma\,|\,\mathbf{u}(\gamma)<0\})$.
   (c) *If* $\mathbf{u}\in V_{1,k}$ *for* $k\geqslant 4$, *then no path partition supports* $\mathbf{u}$. *Furthermore,* $\mathbf{u}$ *satisfies* (B3) *and* (B4) *if and only if* $\mathbf{u}$ *is concentrated.*

(4) *Each* $(Q,\mathbf{u}_Q),(P,\mathbf{u}_P)$, *and* $(\pi_{PP''},\mathbf{u}_{PP''})$ *is an exclusive pair.*

Proof. For (1) use Lemmas 6.2(3), 6.3(3) and 6.3(5). For (2) use Lemma 6.2(5). For (3) use Lemma 6.4. For (4) use Lemmas 6.2(4) and 6.3(4). □

7. Further comments

In this section we briefly compare the cases considered and show that certain convenient situations occur in general. We quote without proof the following result which is a special case of Corollary 2.3 of [2].

Lemma 7.1. *Let Γ be a connected graph with largest two eigenvalues $\theta_0>\theta_1$, $\mathbf{v}$ an eigenvector for θ_0 with all entries positive and $\mathbf{u}$ an eigenvector for θ_1. Then for any $t\geqslant 0$, the induced graph on $\{\gamma \mid \mathbf{u}(\gamma)+t\mathbf{v}(\gamma)\geqslant 0\}$ is connected.*

Lemma 7.2. *If Γ is a regular connected graph and $\mathbf{u}\in V_1$ is an eigenvector for θ_1, the next to largest eigenvalue, then*

(1) *For $t\geqslant 0$, the induced graph on either of the sets $\{\gamma \mid \mathbf{u}(\gamma)\geqslant -t\}$ and $\{\gamma \mid t\geqslant \mathbf{u}(\gamma)\}$ is connected.*
(2) *Suppose that $\mathbf{u}$ is supported by a path partition π. Let $\mathbf{u}(\gamma)=a_i$ for $\gamma\in C_i$ and assume $a_0\geqslant a_\rho$. Then $a_0>0>a_\rho$ and π strongly supports $\mathbf{u}$.*

Proof. (1) follows from applying the previous lemma with $\mathbf{v}=\mathbf{1}$ to $\mathbf{u}$ and to $-\mathbf{u}$. Since $\mathbf{u}$ is orthogonal to $\mathbf{1}$, it has some positive entries and some negative entries. Apply (1) with $t=0$ to see that a_0 and a_ρ are not both nonnegative and not both nonpositive. This means that $a_0>0>a_\rho$. Suppose $a_i>0$ or $a_{i-1}>a_i=0$. Because θ_1 is less than the common vertex degree of Γ, some vertex $\alpha\in C_i$ is adjacent to a vertex with $\mathbf{u}(\beta)<a_i$. Thus $a_0>a_1>\cdots>a_j>a_{j+1}$ where j is the smallest index such that $a_j\geqslant 0>a_{j+1}$. Similarly $a_k>a_{k+1}>\cdots>a_\rho$ where k is the largest index with $a_k>0\geqslant a_{k+1}$. If we show that $k\leqslant j$ we are done. Let $0>-t>a_{j+1}$. Then $\{\gamma \mid \mathbf{u}(\gamma)\geqslant -t\}$ includes C_k and C_j but not C_{j+1}. Since the induced graph on this set is connected, $k\leqslant j$ as desired. □

Remark. The results of this paper strongly depend on the fact that we are considering $\mathbf{u}\in V_i$ for $i=1$. If π is a path partition supporting $\mathbf{u}\in V_1$, then π strongly supports one of $\mathbf{u},-\mathbf{u}$ and π^* strongly supports the other. This need not happen in V_i for $i>1$. Because it did for $i=1$, we were able to find exclusive pairs which are essentially bijections between path partitions and unit vectors of interest.

A vector $\mathbf{u}\in\mathbb{R}^{\mathscr{V}}$ might satisfy some of the following properties, each of which implies the next. In $J(n,d)$ we have seen that all seven conditions are equivalent for $\mathbf{u}\in V_1$:

(1) $\mathbf{u}$ is supported by an equitable distance partition.
(2) $\mathbf{u}$ is supported by a distance partition.
(3) $\mathbf{u}$ is supported by a path partition.
(4) $\mathbf{u}$ is supported by a partition π such that Γ_π is a tree.
(5) every cycle of Γ is balanced by $\mathbf{u}$.
(6) $\mathbf{u}$ satisfies (B3) and (B4).
(7) $\mathbf{u}$ satisfies (B3).

We now give examples to show that (5) does not imply (3) and that none of (2), (3), (4), (6), or (7) implies the previous property, even under the stronger assumption that $\mathbf{u} \in V_1$. We also give an example of $\mathbf{u} \notin V_1$ which satisfies (5) but not (4). In $H(d,n)$ for $n > 2$, (6) $\Rightarrow$ (2), but there are boxes B such that $\mathbf{u}_B$ satisfies (2) but not (1). In $K_{d\times n}$ for $n > 2$, (2) $\Rightarrow$ (1) but there are independent pairs (P, P'') such that $\mathbf{u}_{PP''}$ satisfies (3) but not (2). Also, for $n > 3$, there are concentrated vectors $\mathbf{u} \in V_{1,4}$ which are supported by a star but not a path. Hence, they satisfy (4) but not (3).

The previous example shows that (5) does not imply (3). In all three cases studied, (5) implies (4) for $\mathbf{u} \in V_1$. In the case that Γ is a 10-cycle, a vector $\mathbf{u} \in \mathbb{R}^{\mathscr{V}}$ of type (a,b,c,d,a,c,b,a,d,c) with a,b,c,d distinct balances the unique cycle yet is not supported by a partition π such that Γ_π is a tree. Of course no eigenvector is of this type. For this Γ there are vectors in V_1 with all entries distinct. They vacuously satisfy (6) but not (5). Similarly, $H(2,d)$ and $K_{d\times 2}$ have no triangles and have $\mathbf{u} \in V_1$ which satisfy (7) but not (6). In practice, we simply listed all the cycle conditions used. If needed, we would use balance conditions on larger cycles.

We leave the following questions unanswered, there are relevant results in [2]: Given that Γ is a connected regular graph and π is a partition supporting an eigenvector $\mathbf{u}$ for the next to largest eigenvalue

1. What trees can occur as Γ_π?
2. Does (5) imply (4)?

References

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